Close *Thurs*: HW_4A,4B,4C (6.4,6.5)
Close next Wed: HW_5A, 5B, 5C (7.1,7.2,7.3)

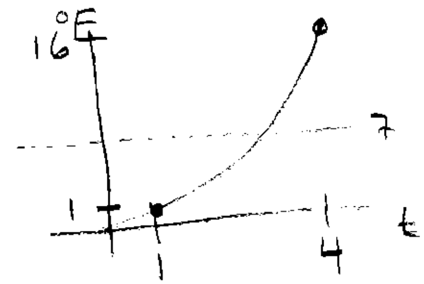
6.5 (~~continued~~) Average Value

The average *y*-value of *y = f(x)* from *x = a* to *x = b* is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Entry Task:
The formula for the temperature of a particular object is $T(t) = t^2$ degrees Fahrenheit where *t* is in hours.
Find the average temperature from $t = 1$ to $t = 4$ hours.

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{4-1}\int_1^4 t^2 dt$$
$$= \frac{1}{3}\left[\frac{1}{3}t^3\Big|_1^4\right]$$
$$= \frac{1}{9}(4^3 - 1^3)$$
$$= \frac{1}{9}(64-1)$$
$$= \boxed{7}\ °F$$



The mean value theorem for integrals:
If $f(x)$ is continuous on from $x = a$ to $x = b$, then there is at least one value $x = c$ at which

$$f(c) = f_{ave}.$$

Example:
Using $T(t) = t^2$ from $t = 1$ to $t = 4$ again. Find a time at which the temperature is exactly equal to the average value.

$$t^2 \overset{?}{=} 7$$

$$\boxed{t = \sqrt{7} \text{ hours}}$$

Average Value Derivation

The average value of the *n* numbers:

$$y_1, y_2, y_3, \ldots, y_n$$

is given by

$$\frac{y_1+y_2+y_3+\cdots+y_n}{n} = y_1\frac{1}{n} + \cdots + y_n\frac{1}{n}.$$

Ex) Four test score

60, 70, 80, 90

$$\text{AVERAGE} = \frac{60+70+80+90}{4} = 75$$

Goal: We want the average value of **all** the y-values of some function y = f(x) over an interval x = a to x = b.

Ex) $f(t) = t^2 \quad 1 \le t \le 4$

n = 6 subdivisions

$$\Delta t = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2} \Rightarrow 6 = \frac{3}{\Delta t}$$

$t_0 = 1,\ t_1 = 1.5,\ t_2 = 2,\ t_3 = 2.5,\ t_4 = 3,\ t_5 = 3.5,\ t_6 = 4$

$y_1 = f(t_1) = (1.5)^2,\ y_2 = (2)^2, \ldots, y_6 = (4)^2$

$$\text{AVE} \approx \frac{\left((1.5)^2 + (2)^2 + \cdots + (4)^2\right)}{6}$$

$$= \frac{1}{6}\left((1.5)^2 + (2)^2 + \cdots + (4)^2\right)$$

$$= \frac{\Delta t}{3}\left((1.5)^2 + (2)^2 + \cdots + (4)^2\right)$$

$$= \frac{1}{3}\left((1.5)^2\Delta t + 2^2\Delta t + \cdots + 4^2\Delta t\right)$$

$$= \underbrace{\frac{1}{3}}_{b-a}\left(f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_6)\Delta t\right)$$

$$\text{AVE VALUE} = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

Derivation:

1. Break into *n* equal subdivisions

$$\Delta x = \frac{b-a}{n}, \text{ which means } \frac{\Delta x}{b-a} = \frac{1}{n}$$

2. Compute *y*-value at each tick mark

$$y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$$

3. $\text{Ave} \approx f(x_1)\frac{\Delta x}{b-a} + \cdots + f(x_n)\frac{\Delta x}{b-a}$

$$\text{Average } \approx \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

Thus, we can define

$$\text{Average} = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

Which means the exact average y-value of y = f(x) over x = a to x = b is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex) $\ldots$, $0 \le x \le 2$

$n = 6$

SEE PREVIOUS PAGE FOR AN EXAMPLE

7.1 Integration by Parts

Goal: We will reverse the product rule.

Before we start, add these to your basic list of integrals:

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{1}{\mathrm{ax+b}}dx = \frac{1}{a}\ln|ax+b| + C$$

Ex)

$$\int \sin(5x)dx = \frac{-1}{5}\cos(5x) + C$$

CHECK

$$\int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + C$$

CHECK

$$\int \frac{1}{4x+3}dx = \frac{1}{4}\ln|4x+3| + C$$

CHECK

Derivation of Integration By Parts

The <u>product rule</u> says:

$$u(x)v'(x) + v(x)u'(x) = \frac{d}{dx}(u(x)v(x))$$

which can be written as

$$\int u(x)v'(x)dx + \int v(x)u'(x)dx = u(x)v(x)$$

Writing this in terms of the differentials:

$$dv = v'(x)dx \text{ and } du = u'(x)dx$$

we have

$$\int u\,dv + \int v\,du = uv$$

which we rearrange to get

Integration by Parts formula:

$$\int u\,dv = uv - \int v\,du$$

Ex)

$$3x \cdot \cos(x) + 3\sin(x) = \frac{d}{dx}(3x\sin(x))$$

$$\int 3x\cos(x)\,dx + \int 3\sin(x)\,dx = 3x\sin(x)$$

$$+c$$

$$\Rightarrow \int 3x\cos(x)\,dx = 3x\sin(x) - \int 3\sin(x)\,dx$$

Example:

$$\int x\cos(8x)dx$$

$u = x \qquad dv = \cos(8x)dx$

$du = dx \qquad v = \frac{1}{8}\sin(8x)$

Step 1: Choose *u* and *dv*.
Step 2: Compute *du* and *v*.
Step 3: Use formula (and hope)

$= \frac{1}{8}x\sin(8x) - \int \frac{1}{8}\sin(8x)dx$!!

$= \frac{1}{8}x\sin(8x) - \frac{1}{64}(-\cos(8x)) + C$

$= \frac{1}{8}x\sin(8x) + \frac{1}{64}\cos(8x) + C$

CHECK!!!

$\frac{1}{8}\sin(8x) + x\cos(8x) - \frac{1}{8}\sin(8x)$

SAME ✓✓✓

Example:

$$\int x^2 \ln(x)\, dx$$

$u = \ln(x)$ $\quad dv = x^2 dx$

$du = \frac{1}{x} dx$ $\quad v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3} x^2 dx$

$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$

CHECK!

$x^2 \ln(x) + \frac{1}{3}x^3 \cdot \frac{1}{x} - \frac{1}{3}x^2$

SAME ✓✓

Example:

$$\int_1^e x^2 \ln(x)\, dx$$

$u = \ln(x) \quad dv = x^2\, dx$

$du = \frac{1}{x} dx \quad V = \frac{1}{3}x^3$

$$\frac{1}{3}x^3 \ln(x)\Big|_1^e - \int_1^e \frac{1}{3}x^2\, dx$$

$$\left(\left(\frac{1}{3}e^3\ln(e)\right) - 0\right) - \frac{1}{9}\left(x^3\Big|_1^e\right)$$

$$\frac{1}{3}e^3 - \frac{1}{9}\left(e^3 - 1\right)$$

$$\frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$-\frac{2}{9}e^3 + \frac{1}{9}$$

$$\frac{1}{9}(1 - 2e)$$

NOTE: FROM PREVIOUS PAGE

$$\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3\Big|_1^e$$

$$= \left(\frac{1}{3}e^3\ln(e) - \frac{1}{9}e^3\right) - \left(0 - \frac{1}{9}\right)$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$= -\frac{2}{9}e^3 + \frac{1}{9}$$

$$= \frac{1}{9}(1 - 2e^3)$$

Notes:

1. The symbols *u* and *v* **never** appear in the integration. They are just locations in the formula (no variables are changing, this is not substitution).

2. *u* and *dv* completely split up the integrand. **Once you chose *u*, then *dv* is everything else.**

3. The goal is to make $\int v\, du$ "nicer" than $\int u\, dv$
   (a) Pick u = "something that gives a derivative that is simpler than the original u"
   (b) Pick dv = "something that you can integrate"
   (c) And hope "vdu" is something in our table!

Example:

$$\int x^2 e^{x/2} dx$$

$u = x^2 \quad dv = e^{\frac{1}{2}x} dx$

$du = 2x dx \quad v = 2e^{\frac{1}{2}x}$

$= 2x^2 e^{\frac{1}{2}x} - \int 4x e^{\frac{1}{2}x} dx$

$u = 4x \quad dv = e^{\frac{1}{2}x} dx$

$du = 4dx \quad v = 2e^{\frac{1}{2}x}$

SAME ✓✓✓

$= 2x^2 e^{\frac{1}{2}x} - \left(8x e^{\frac{1}{2}x} - \int 8 e^{\frac{1}{2}x} dx\right)$

$= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 8 \int e^{\frac{1}{2}x} dx$

$= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16 e^{\frac{1}{2}x} + C$

CHECK!

$4x e^{\frac{1}{2}x} + x^2 e^{\frac{1}{2}x} - 8 e^{\frac{1}{2}x} - 4x e^{\frac{1}{2}x} + 8 e^{\frac{1}{2}x}$

Example:

$$\int e^x \cos(x)\, dx$$

$u = e^x \quad dv = \cos(x)dx$

$du = e^x dx \quad v = \sin(x)$

$$= e^x \sin(x) - \int e^x \sin(x) dx$$

$u = e^x \quad dv = \sin(x)dx$

$du = e^x dx \quad v = -\cos(x)$

$$= e^x \sin(x) - \left(-e^x \cos(x) - \int -e^x \cos(x) dx\right)$$

SAME ✓✓

$$\Rightarrow \quad \int e^x \cos(x) = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$$

$\Rightarrow$ (up to a constant) $\quad 2\int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) + C_0$

$$\int e^x \cos(x) dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C \qquad C = \tfrac{1}{2} C_1$$

$$= \frac{1}{2} e^x \left(\sin(x) + \cos(x)\right) + C$$

CHECK!!!

$$\frac{1}{2} e^x \cancel{\sin(x)} + \underbrace{\frac{1}{2} e^x \cos(x) + \frac{1}{2} e^x \cos(x)}_{e^x \cos(x)} - \cancel{\frac{1}{2} e^x \sin(x)}$$

Example:

$$\int \sin^{-1}(x)\,dx$$

$u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $\quad v = x$

$$= x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

NOW WHAT?!?
SUBSTITUTION!

$u = 1-x^2$

$du = -2x\,dx$

$\frac{1}{-2x}du = dx$

$$= x\sin^{-1}(x) - \int \frac{\cancel{x}}{\sqrt{u}}\frac{1}{-2\cancel{x}}du \qquad u^{-1/2}$$

$$= x\sin^{-1}(x) + \frac{1}{\cancel{2}}\,\cancel{2}\,u^{1/2} + C$$

$$= \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

$\boxed{\text{ASIDE}}$ IF YOU FORGOT $\frac{d}{dx}\left(\sin^{-1}(x)\right)$

HERE IS HOW WE DERIVED IT IN MATH 124.

$y = \sin^{-1}(x)$

$\Rightarrow \sin(y) = x$

$\frac{d}{dx}\left[\sin(y) = x\right]$

$\Rightarrow \cos(y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$

$\boxed{\text{From } 124}$

Now $\cos^2(y) = 1 - \sin^2(y)$

$\Rightarrow \cos(y) = \pm\sqrt{1-\sin^2(y)}$

ON DOMAIN $-\pi/2 \le x \le \pi/2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sin(y))^2}}$ $\quad x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$